3

Superconductivity

A conductor is a material that can transport electric charge. A typical example of a conductor is a metal, such as copper, iron, or gold. These materials are formed by a somewhat regular lattice of positively charged ions and a sea of mobile electrons. Subject to an external force, the valence electrons will accelerate toward the regions of lowest potential energy, creating an electrical current. As opposing agent to this acceleration, the electrons experience a friction due to the collisions with ions in the lattice. A typical electron will move an average distance, called the *mean free electron path*, before it hits an ion, exchanges kinetic energy with it, and randomly changes both the velocity and direction of motion.

What I described in the last paragraph is the usual microscopic model for Ohmic resistance. At the macroscopic level, it implies that, in order to sustain a current I flowing through a conductor, we have to continously exert a force on the charge carriers, establishing a potential energy difference or *voltage* V between two points of the conductor. The larger the current we wish to establish, the faster the material dissipates energy and the larger the voltage V required. The proportionality constant between current I and voltage V is the device's resistance R:

$$V = I \times R. \tag{3.1}$$

The resistance R is an extensive property that depends on the size of the material through which charge carriers propagate, as well as the rate of collisions that we mentioned before, the so-called *resistivity* ρ.

From our microscopic interpretation of the resitivity ρ, it is obvious that there will be many contributions to this value. Part of the resistivity will be due to *intrinsic* sources. These include lattice defects – dislocations, boundaries between grains in the material, different orientations in the crystal – and impurities or atoms from different materials that penetrate the conductor. In addition to these, we also have to consider the influence of temperature. When the metal heats up, the ions move faster and farther away from their equilibrium positions in the lattice. This increases both

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Table 3.1. *Critical temperatures of various superconducting materials.*

Element or compound	Symbol	SC type	T_c	Gap 2Δ
Aluminum	Al	Type I	1.175 K	82.2 GHz
Lead	Pb	Type I	7.20 K	660 GHz
Mercury	Hg	Type I	4.15 K	399 GHz
Niobium	Nb	Type II	9.2 K	738.5 GHz
Yttrium barium copper oxide	$YBa_2Cu_xO_y$	High-T_c	70–90 K	–

the probability of collisions with electrons as well as the amount of energy that they can exchange. When we combine both effects, we typically find an experimental fit of the form

$$\rho \sim \rho_0[1 + \alpha \max\{T - T_0, 0\}], \tag{3.2}$$

with a temperature-dependent contribution and an intrinsic plateau ρ_0 that depends on the material and even on the specific sample.

The study of the temperature-dependent resistance in metals experienced a breakthrough with the discovery of liquid helium by Kamerlingh Onnes in 1908. One of the first applications of liquified helium was to study the conductivity of metals (mercury, tin, and lead) under temperatures $\sim$4.2 K where the thermal contribution to the resistivity should be negligible. In 1911, Kamerlingh Onnes submerged a wire of mercury in helium and observed that the resistivity of the metal suddenly dropped to zero. In other words, the previous plateau disappeared, $\rho_0 \simeq 0$, even though nothing special had been done to improve the material's purity and lattice perfection. After these initial discoveries, further materials were shown to enter this new *superconducting* phase at sufficiently low temperatures, summarized in Table 3.1.

Later studies have shown that a superconducting material cannot be just characterized as a conductor without resistance. As we will soon see, superconductivity is an intrinsically quantum effect that has other unexpected consequences:

- The first one is the possibility of establishing *persistent currents*. If we build a superconducting ring and induce a current, for instance by passing a magnet through the loop or using an external inductor, this current can persist forever without any energy penalty on the material.
- The current that flows around a hole or in a loop can only take certain discrete values that are compatible with the quantization of the magnetic flux that traverses the loop. This *fluxoid quantization* happens in units of the *flux quantum* $\Phi_0 = h/2e$, a universal magnitude governing the operation of SQUIDs (Section 4.7) and superconducting qubits (Chapter 6).

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- Superconductors exhibit another surprising phenomenon known as the *Meissner effect*, which is the ability of the superconductor to "expel" magnetic fields. If we take a superconducting object and switch on a magnetic field, the material develops superconducting currents on the surface such that they cancel the magnetic fields inside the object. This phenomenon is similar to how charges on the surface of a conductor arrange to cancel all *electric* fields inside the bulk and lays at the heart of different levitation experiments with magnets and superconductors.
- Finally, superconductivity also leads to counterintuitive behavior in superconductor–insulator–superconductor interfaces. The quantum nature of the superconducting charge carriers allows them to tunnel through thin insulating barriers, establishing currents that behave in unexpected ways in presence of dc and ac potentials. This *Josephson effect* makes it possible to develop *Josephson junctions*, a nonlinear inductor that makes superconducting circuits useful for quantum technology applications.

This phenomenology, which is universal across the board of all superconducting materials, can be explained with a mesoscopic theory that combines quantum mechanics and electromagnetism. However, not all superconducting materials are identical: They can be distinguished by other properties, such as the conditions under which superconductivity appears or is destroyed.

- *Type I superconductors* is the denomination for the first family of superconducting materials that were discovered. This includes most elementary metals, such as mercury, tin, or aluminum. Superconductivity manifests at low *critical temperatures,* between 1–4 K, and is destroyed at relatively low critical magnetic fields. Above such fields, the Meissner effect abruptly disappears and the material experiences a first-order phase transition into an ordinary metal.
- *Type II superconductors* includes niobium and a plethora of alloys with larger critical temperatures and superconducting gaps. These materials survive stronger magnetic fields through the creation of magnetic vortices: thin "tubes" of ordinary metal that allow the field to cross the material, while the rest of the electrons remain in a superconducting state.
- *High-Tc superconductors* is yet another family of superconducting materials, discovered late in the twentieth century. These superconductors are rare-earth ceramic alloys with a complex, quasi-two-dimensional structure that allows superconductivity at temperatures above nitrogen's boiling point (70 K). Since we must work at lower temperatures to engineer microwave quantum circuits (see Section 4.1.1), and since these ceramic materials are difficult to fabricate, they are not very interesting for superconducting quantum information technology.

After this brief overview of superconducting materials, we will introduce both a microscopic interpretation of superconductivity, as well as a macroscopic theory – the London theory or macroscopic wavefunction model – that explains much of the superconducting phenomenology. A distilled version of this model will be the basis in Chapter 4 to develop an effective theory of superconducting circuits.

3.1 Microscopic Model

Which mechanism allows some metals to become superconductors? This is a Nobel Prize–winning question that was answered by physicists John Bardeen, Leon Cooper, and John R. Schrieffer in 1957 (Bardeen et al., 1957a,b). Together, they explained many of the superconducting phenomena mentioned previously, as well as other many-body properties of the superconducting materials, which include the following:

- The evidence of a phase transition and of some type of energy gap to break superconductivity, as suggested by the existence of a critical temperature T_c and a critical magnetic field above which superconductivity disappears.
- The exponential decrease of the superconductor's heat capacity with temperature $C \propto \exp(-1.5T_c/T)$. This property was consistent with a many-body theory in which there is an energy gap, a minimum excitation energy per particle of order $\simeq 1.5k_B T_c$.
- Further evidence of some excitation gap, as provided by the electromagnetic absorption spectrum: the minimum photon energy required to locally excite or break the superconducting state lays somewhere around $2 \times 1.5 \times k_B T_c$.
- Finally, the magnetostatic properties of the superconductor, including superconducting currents and the Meissner effect could be explained by introducing a *coherence length* – i.e., quantum correlations at short distances. As it is now well known in condensed matter physics, the existence of finite coherence lengths is usually an indicator of a gapped model.

The answer by Bardeen and collaborators is known as the *BCS theory*. This theory proposes that the superconductor is actually a Bose–Einstein condensate of charge carriers. As it was already known from studies of Bose–Einstein condensation and early models of ^{4}He superfluidity, a weakly interacting Bose–Einstein condensate can support superfluid currents that never stop and that are immune to small imperfections, impurities, and collisions that do not carry too much energy. In order to justify the existence of Bose–Einstein condensation, the BCS theory introduces an effective attraction between the metal's valence electrons with opposite spin. This attraction is mediated by the phonons of the crystalline structure that forms the metal. At low temperatures, it gives rise to the BCS instability, in which the

Fermi theory breaks down, and electrons join into stable bound particles called *Cooper pairs* (Cooper, 1956). Because of the spin-statistics connection, the pair of two bound electrons is a particle with a bosonic statistics, which condense into a superfluid state.

The BCS theory provides a very elegant and also very straightforward framework for studying the superconductor, with only a few parameters that account for all the physics. The main parameter is the *BCS or superconducting gap*, usually denoted by $\Delta(\mathbf{k})$. The gap is the binding energy of every two electrons that form a pair. It depends mildly on the electron's momenta, and the smallest value $\Delta(0)$ explains many quantiative properties of the superconducting phase and the phase transtion. For instance, as we increase the temperature, we can expect that processes in which pairs are broken become relevant, destroying superconductivity. The quantitative answer is a bit more complicated, as the gap itself depends on temperature $\Delta(T) \simeq 1.74\Delta(0)\sqrt{1 - T/T_c}$, and the critical temperature $T_c \simeq \Delta(0)/1.76k_B$ is the point at which pairing becomes energetically trivial.

The superconducting gap also explains some features in the interaction of the superconductor with electromagnetic fields. A superconductor can absorb photons through two different mechanisms. Low-energy microwaves in the range of 1–20 GHz can excite plasmons of the charged superfluid. These processes create quantum excitations that behave very much like photons (see Chapter 5) or like artificial atoms (see Chapter 6), and which we can route, confine, and operate using superconducting circuits. The second mechanism involves stealing a Cooper pair from the condensate and breaking it into two separate electrons. The energy required for this is $\hbar\omega \simeq 2\Delta(0) \simeq 3.52k_BT$. For the case of the widely used material in superconducting circuits, aluminum, this energy lays in the range of 100 GHz. Therefore, we can suppress this type of event by sufficiently cooling our circuits and isolating them from the environment, with filters that prevent the injection of highly energetic photons.

The BCS theory has other important consequences, including studies of heat capacity, impurities, quasiparticle excitations, Andreev states and normal superconductor interfaces. Overall, this theory applies very well to type I superconductors, and to some extent to type II, but it does not explain high-temperature superconductivity.

Fortunately, we are not so much interested in complex superconducting materials or sophisticated excitations. Rather, we would like an effective model of the superfluid condensate in the simple materials, Al or Nb, which are used in the quantum circuit experiments. As explained by Gor'kov (1959), the BCS model of superconductivity predicts an effective nonlinear theory for the condensate order parameter. This is the Ginzburg–Landau model or, in the simplified linear version that we introduce here, the macroscopic wavefunction model

(Orlando, 1991). Without many complications, this intuitive model will provide a solid and approachable foundation to the engineering of quantum circuit in later chapters.

3.2 Macroscopic Quantum Model

The BCS model for superconductivity proposes that electrons group into a larger unit, the Cooper pair, that has the properties of both being a charged particle, with charge $q = -2e$, as well as being a boson. The bosonic nature of the particle is what makes it possible for all Cooper pairs to condense, sharing a common superfluid state that is insensitive to defects in the material and any other drag force. The theory that we are about to explain is very similar to other models that have been put forward and successfully used, for instance, in the study of weakly interacting Bose–Einstein condensates of alkali atoms (Pitaevskii and Stringari, 2016), and even BCS superfluids built from fermionic atoms. Our formulation of the theory follows closely Orlando (1991), a book we encourage you to read for a better understanding of superconducting properties, magnetostatics, and other interesting phenomenology.

The macroscopic wavefunction theory is based on the assumption that the many-body state is described by a collective wavefunction that is a product state of the same wavefunction for each of the N Cooper pairs:

$$\Xi(\mathbf{x}_1, \mathbf{x}_2 \ldots \mathbf{x}_N; t) = \xi(\mathbf{x}_1, t)\xi(\mathbf{x}_2, t) \cdots \xi(\mathbf{x}_N, t). \tag{3.3}$$

This type of macroscopic accumulation of particles into the same quantum state is what we expect from a Bose–Einstein condensate well below its critical temperature. However, we are also allowing for this accumulation, which is typically a property of a ground state, to also describe the dynamics of the collective system in time, as it reacts to external perturbations from electromagnetic fields, currents, etc. This is a conceptual extension that is only justified by the agreement with experiments and the exact simulations of small systems.

The macroscopic wavefunction theory leads us to introduce new fields n_s and θ, which respectively describe the charge density and, as we will soon see, the flow of particles:

$$\psi(\mathbf{x}, t) = \sqrt{N}\xi(\mathbf{x}, t) \simeq \sqrt{n_s(\mathbf{x}, t)}e^{i\theta(\mathbf{x}, t)}. \tag{3.4}$$

We typically assume a constant and approximately uniform density of carriers throughout most of the material, $n_s(x, t) \simeq \bar{n}_s$. This assumption is approximately valid in many situations because matter tends toward charge neutrality. It will not apply when we consider the charge trapped on a capacitor or in a superconducting

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island. Those deviations will be studied as perturbative corrections to the background of superconducting particles in the macroscopic theory.

Assuming that the macroscopic wavefunction is a viable model, we now postulate a very general model for its dynamics:

$$i\hbar\partial_t\psi = \left[\frac{1}{2m_s}(-i\hbar\nabla - q_s\mathbf{A})^2 + q_s v(\mathbf{x},t)\right]\psi. \tag{3.5}$$

This model is inspired by the Schrödinger equation for a charged particle moving in an electromagnetic field with scalar and vector potentials $v(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$, respectively. The model introduces two effective parameters m_s and q_s to describe the mass and charge of the Cooper pair. As we have seen, the charge is precisely known:

$$q_s = -2e = -2 \times 1.60217662 \times 10^{-19}\ \text{C}. \tag{3.6}$$

However, $m_s = 2m_e^*$ contains the effective mass of the electrons moving through the solid lattice m_e^*, which depends on the band structure and has to be determined experimentally for each material.

3.3 Superfluid Current

From the previous equation, we can already obtain two important properties that we need for studying real superconducting circuits. The first property is the charge distribution, which is given by

$$\rho(\mathbf{x},t) = q_s|\psi(\mathbf{x},t)|^2. \tag{3.7}$$

This superfluid charge includes a very large background that compensates the charge of the ions structuring the lattice of the metal or alloy. From the point of view of circuit theory, it is more interesting to work with the *superfluid current*, a vector field $\mathbf{J}(\mathbf{x},t)$ describing the flow of charges. The evolution of the electric charge Q confined in a volume Ω, is related to the supercurrent flowing across the boundary of that volume $\partial\Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}t}Q = \int_\Omega \partial_t\rho\,\mathrm{d}^3x = -\int_{\partial\Omega}\mathbf{J}\cdot\mathrm{d}\mathbf{n}. \tag{3.8}$$

Here $\mathbf{n}$ is the unit vector normal to the surface $\partial\Omega$ at each point of the boundary. The same physics is described by the continuity equation:

$$\partial_t\rho = -\nabla\cdot\mathbf{J}. \tag{3.9}$$

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As Fritz London conjectured, the superfluid current may be derived from the Schrödinger equation (3.5) as a combination of the macroscopic wavefunction current and the electromagnetic field:

$$\mathbf{J} = q_s \times \mathrm{Re}\left\{\psi^* \left[\left(-i\frac{\hbar}{m_s}\nabla - \frac{q_s}{m_s}\mathbf{A}\right)\psi\right]\right\}. \tag{3.10}$$

This combination explain the fluxoid quantization and the Meissner effect, as described by London et al. (1935).

Note that we can obtain the *electric current intensity* of a circuit I by integrating the charge current $\mathbf{J}(\mathbf{x}, t)$ across any section S of any cable in the circuit:

$$I(t) = \iint_S \mathbf{J}(\mathbf{x}, t) \cdot \mathrm{d}\vec{\mathbf{S}}. \tag{3.11}$$

The current intensity can be different at different points of a large superconducting circuit. However, the conservation of charge – Cooper pairs are not destroyed in our simple, conservative model – implies that the current coming into a superconducting element must balance with the current going out into other circuit elements. This will be key in our analysis of circuits and derivation of quantitative models in Chapter 4.

3.4 Superconducting Phase

The superconducting wavefunction contains information about the charge distribution and the electrical current. We will now argue that most of the information is actually hidden in the phase of the wavefunction. We will also relate this phase to a macroscopically observable quantity, the flux.

The first statement is rather obvious. We have already discussed that the density of charged particles must be a rather uniform property, dependent only on the properties of the material – i.e., how many electrons the atoms donate to the conduction band where Cooper pairs are formed. If we assume that $n_s(\mathbf{x}, t) = |\psi|^2$ is constant and uniform, currents are divergence-free:

$$\nabla \cdot \mathbf{J} = 0, \tag{3.12}$$

and all information about the superconductor must actually reside in the phase of the wavefunction. Indeed, working with (3.10), we obtain

$$\mathbf{J} = q_s n_s \left[\frac{\hbar}{m_s}\nabla\theta - \frac{q_s}{m_s}\mathbf{A}\right]. \tag{3.13}$$

The second statement is more subtle. Let us assume the *Coulomb gauge* $\nabla \cdot \mathbf{A} = 0$, and specialize the macroscopic quantum model (3.5) for a wavefunction with uniform density. The Coulomb gauge implies $\Delta\theta = 0$, and

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$$-\hbar\partial_t\theta \simeq \frac{1}{2n_s}\Lambda\mathbf{J}^2 + q_s v. \tag{3.14}$$

Notice the new constant, the isotropic London coefficient[1] $\Lambda = m_s/q_s^2 n_s$.

In the absence of currents, $\mathbf{J} = 0$, (3.14) becomes the so-called *phase-voltage* relation

$$\partial_t\theta \simeq -\frac{q_s}{\hbar}v. \tag{3.15}$$

This relation is an obvious consequence of unitary evolution. For quasi-stationary states, the wavefunction remains constant up to a global phase, determined by the energy of the system $\psi(\mathbf{x},t) = \exp(-iEt/\hbar)\psi(\mathbf{x},0)$. Since the energy of the charged particle in a potential is $E = q_s v$, we obtain $\partial_t\theta = -E/\hbar = -q_s v/\hbar$. The problem with relation (3.15) is that it is not gauge invariant – it is only valid in the Coulomb gauge – and it has been derived under the condition of no persistent currents, $\mathbf{J} = 0$. We have to complete our derivation to regard more general conditions!

3.5 Gauge-Invariant Phase

In order to correct (3.15), we will separate the superconducting phase into a term that is always the same, and a contribution that depends on our choice of electromagnetic gauge. The *gauge-invariant phase* $\varphi(\mathbf{x},t)$ is defined by removing the contribution of the vector potential, taking as reference one (arbitrary) location of the superconductor $\mathbf{x}_0$:

$$\theta(\mathbf{x},t) - \theta(\mathbf{x}_0,t) = \varphi(\mathbf{x},t) - \varphi(\mathbf{x}_0,t) + \frac{q_s}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{A}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{l}. \tag{3.16}$$

In the definition of $\varphi(\mathbf{x},t)$, the choice of path from $\mathbf{x}_0$ to $\mathbf{x}$ is arbitrary, but (i) it must be unique for each point $\mathbf{x}$, (ii) it must be continuous, (iii) all paths must remain in the superconductor, and (iv) they must not cross each other.[2] Except for a set of points of zero measure – the discontinuities of φ – we can define a gauge-invariant wavefunction:

$$\psi_{\mathrm{GI}}(\mathbf{x},t) = e^{-i\frac{q_s}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}}\mathbf{A}(\mathbf{r},t)\cdot\mathrm{d}\mathbf{l}}\psi(\mathbf{x},t) = e^{i\varphi(\mathbf{x},t)}\sqrt{n_s(x,t)}, \tag{3.17}$$

[1] As mentioned before, superconducting currents shield magnetic fields out of the material. The London coefficient is related to the penetration depth of magnetic fields into the superconductor, a fact that can also be derived from this theory (Orlando, 1991).

[2] This choice of path is one of the essential steps in working with any superconducting circuit, as we will see in Chapter 4. However, in that chapter we will assume quasi-one-dimensional structures, where it is easy to believe that such paths do exist, at least in the form $\mathbf{x}_l = \mathbf{x}l + \mathbf{x}_0(1-l)$ for $l \in [0,1]$.

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which satisfies a Schrödinger equation without vector potential:

$$i\hbar\partial_t\psi_{\mathrm{GI}} = \left[\frac{1}{2m_s}(-i\hbar\nabla)^2 + q_s v(\mathbf{x},t)\right]\psi_{\mathrm{GI}}. \tag{3.18}$$

We now reach a gauge-independent relation between the phase and the electric field:

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(\mathbf{x},t) &= \frac{q_s}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}}\left(-\nabla v - \frac{\partial}{\partial t}\mathbf{A}\right)\cdot \mathrm{d}\mathbf{l} \qquad (3.19)\\ &= \frac{q_s}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}}\mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{l} = \frac{q_s}{\hbar}\left[V(\mathbf{x}_0) - V(\mathbf{x})\right].\end{aligned}$$

This gauge-invariant phase is related to the voltage difference V across the superconducting circuit,[3] defined as the energy required to transport a unit of charge from $\mathbf{x}_0$ to $\mathbf{x}$. It is convenient to refer the voltage to $V(\mathbf{x}_0) := 0$ and work instead with the electric *flux*, defined up to an irrelevant offset as

$$\phi(\mathbf{x},t) = \int_0^t V(\mathbf{x},\tau)\mathrm{d}\tau. \tag{3.20}$$

Introducing also the *magnetic flux quantum*,

$$\Phi_0 = \frac{h}{|q_s|} = \frac{h}{2e} \simeq 2.067833758(46)\times 10^{-15}\,\mathrm{Wb}, \tag{3.21}$$

we identify the gauge-invariant phase with the electric flux

$$\partial_t\varphi(\mathbf{x},t) = \frac{2\pi}{\Phi_0}\partial_t\phi(\mathbf{x},t), \tag{3.22}$$

As we will see in Chapter 4, this identity makes the electric flux one of the two preferred variables when working with superconducting circuits, the other one being the charge q that accumulates on a superconducting element. Another important property of the gauge-invariant phase is that it determines the superconducting current:

$$\mathbf{J} = q_s n\left[\frac{\hbar}{m_s}\left(\nabla\varphi + \frac{q_s}{\hbar}\mathbf{A}\right) - \frac{q_s}{m_s}\mathbf{A}\right] = \frac{\hbar q_s n_s}{m_s}\nabla\varphi. \tag{3.23}$$

This makes sense, because the superconducting current, just like the circuit voltage, cannot depend on the choice of gauge.

[3] V and v are not the same observable. Only the former is gauge independent.

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3.6 Fluxoid Quantization

There is a third consequence of the phase-current relation. This one relates to the allowed values of the current and of the magnetic flux trapped inside superconductors. Let us multiply (3.13) by $\Lambda = m_s/n_s q_s^2$ and integrate it around a closed-loop C around a simply connected region, $C = \partial S$:

$$\oint_C (\Lambda \mathbf{J}) \cdot d\mathbf{l} + \oint_C \mathbf{A} \cdot d\mathbf{l} - \oint_C \frac{\hbar}{q_s} \nabla\theta \cdot d\mathbf{l} = 0. \tag{3.24}$$

We use Stokes's theorem to transform the second line integral into a surface integral over the region S. The contour integral of the phase must produce a multiple of $2\pi \times \hbar/q_s = -\Phi_0$, as otherwise the wavefunction ψ would be discontinuous. This results in the quantization equation

$$\oint_C (\Lambda \mathbf{J}) \cdot d\mathbf{l} + \int_S \mathbf{B} \cdot d\mathbf{S} + \Phi_0 \times m = 0, \quad \text{with } m \in \mathbb{Z}. \tag{3.25}$$

This equation states that the magnetic flux trapped in a loop

$$\Phi_{\text{loop}} = \int_S \mathbf{B} \cdot d\mathbf{S}, \tag{3.26}$$

plus the flux due to the induced supercurrents must be an integer multiple of the magnetic flux quantum (3.21). This is a very powerful result, used by Deaver and Fairbank (1961) to estimate Φ_0 and demonstrate that the superconducting particles $q_s = -2e$ are formed by two electrons.

There is a simpler derivation of fluxoid quantization that is of more interest to us, and which is based on the gauge-invariant phase (3.16). This definition can be reformulated as

$$\nabla\theta = \nabla\varphi + \frac{q_s}{\hbar}\mathbf{A}. \tag{3.27}$$

Integrating around a closed loop, we find

$$\oint_C \nabla\varphi \cdot d\mathbf{l} = 2\pi \times m + \frac{2\pi}{\Phi_0}\Phi_{\text{loop}}. \tag{3.28}$$

Using the phase-voltage relation, this can be rewritten as a condition for the flux differences along different segments of a superconducting circuit, or *fluxoid quantization*:

$$\oint_C \nabla\phi \cdot d\mathbf{l} = \Phi_0 \times m + \Phi_{\text{loop}}. \tag{3.29}$$

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This condition is one of the constituent relations in the theory of superconducting circuits from Chapter 4. Note that unlike the wavefunction, the flux and the gauge-invariant phase need not be continuous, as their definition depends on a choice of paths that we make before studying the circuit. However, our derivation reveals that all choices are consistent, and that the existence of loops in a circuit reduces the number of independent variables, because fluxes are not completely independent from each other.

3.7 Josephson Junctions

The last topic that we cover in this chapter is a simple, yet very powerful device that was postulated by Josephson (1962) and verified experimentally shortly thereafter by Anderson and Rowell (1963). The device in question is called a *tunnel* Josephson junction.[4] It is a superconductor–insulator–superconductor "sandwich," where the insulating area is so thin that it allows quantum tunneling of Cooper pairs. As Josephson predicted, the tunneling of pairs creates unexpected relations between the applied voltage (dc or ac) on the superconducting leads and the intensity that circulates through the junction. We can use the macroscopic quantum model to derive those relations.

Figure 3.1b shows the schematics of a Josephson junction, with three separate regions: two superconducting leads of arbitrary size and an insulating barrier. Our toy model for the junction is a one-dimensional[5] potential barrier of height U_0 and width d that hinders the propagation of the macroscopic wavefunction $\psi(x,t)$.

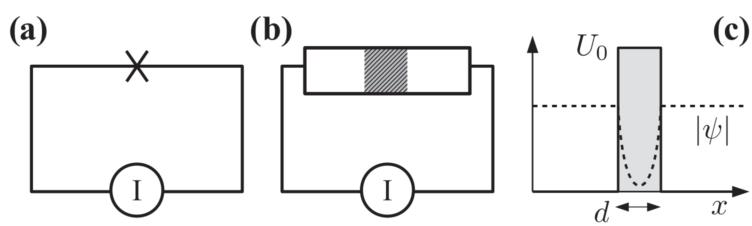

Figure 3.1 (a) Circuit schematics for a Josephson junction connected to an intensity source. (b) The Josephson junction is made of two superconducting leads (white) separated by a thin insulating barrier (gray). (c) We model the junction as a barrier energy U_0, which is thin enough, d, that it allows some quantum tunneling of Cooper pairs.

4 We can obtain a similar physics through other physical devices, such as constrictions, point contacts, and normal interfaces. However, the theoretical picture is simpler in this case.

5 The one-dimensional model is sufficient for describing the type of small junctions that appears in typical superconducting circuits. A more detailed model that takes into account the transverse dimensions is found in Orlando (1991).

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We seek stationary solutions of the Schrödinger equation for the gauge-invariant wavefunction

$$E\psi_{\mathrm{GI}}(x) = \left[\frac{1}{2m_s}(-i\hbar\partial_x)^2 + v_0(x)\right]\psi_{\mathrm{GI}}(x), \tag{3.30}$$

with the potential barrier

$$v_0(x) = \begin{cases} 0, & |x| > d/2 \\ U_0, & \text{otherwise.} \end{cases} \tag{3.31}$$

As a boundary condition, we impose a current intensity I flowing through the junction. Thanks to the relation between current intensity and superconducting current, our boundary condition can be written as

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = J \times A, \tag{3.32}$$

where the proportionality constant A is the area perpendicular to the junction – i.e., the cross-section size.

What do the solutions of the Schrödinger equation look like with these conditions? Since far away from the insulator the current is fixed, we expect plane-wave solutions such that $\partial_x\varphi \propto J$. More precisely, we write

$$\psi_{\mathrm{GI}}(x) \propto \sqrt{n}e^{ikx}, \quad |x| > d/2. \tag{3.33}$$

The sign and magnitude of the superconduction current may be derived from the momentum k of the wavefunction as $J = q_s n_s \hbar k/m_s$. The momentum also k determines the energy of the solution

$$E = \frac{\hbar^2 k^2}{2m_s} = \frac{1}{2n}\Lambda^2 J^2. \tag{3.34}$$

When the current is very small $J \simeq 0$, this energy will lay well below the barrier U_0. Inside the insulator, we will have an equation of the form

$$\partial_x^2\psi_{\mathrm{GI}} = \frac{m_s}{\hbar^2}(U_0 - E)\psi_{\mathrm{GI}} = \frac{1}{\xi^2}\psi_{\mathrm{GI}}, \quad |x| \le d/2, \tag{3.35}$$

which can only be satisfied with exponentially decreasing or increasing solutions, $\psi_{\mathrm{GI}} \propto \exp(\pm x/\xi)$. The final solution then reads

$$\psi_{\mathrm{GI}}(x) = \alpha_+ \cosh(x/\xi) + \alpha_- \sinh(x/\xi), \quad |x| \le d/2, \tag{3.36}$$

with the parameters

$$\alpha_\pm = \sqrt{n_s}\,\frac{e^{i\varphi_L(-d/2)} \pm e^{i\varphi_R(d/2)}}{e^{d/2\xi} \pm e^{-d/2\xi}}. \tag{3.37}$$

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The superfluid current is uniform across the circuit. Along the insulator

$$J = \frac{q_s}{m_s}\mathrm{Re}(-\psi_{\mathrm{GI}}^* i\hbar\partial_x\psi_{\mathrm{GI}}) = J_c \sin[\varphi_R(d/2) - \varphi_L(-d/2)], \tag{3.38}$$

it is proportional to the Josephson junction's critical current

$$J_c = -\frac{q_s\hbar}{m_s\xi}\frac{n_s}{\sinh(d/\xi)} \geq 0. \tag{3.39}$$

Currents above this value cannot be captured by the exponential solution, but instead correspond to plane waves with $E > U_0$.

The previous derivation has shown us that small values of the current have a nonlinear relation with the gauge-invariant phase jump $\delta\varphi = \varphi_{\mathrm{R}} - \varphi_{\mathrm{L}}$ across the junction:

$$I = I_c \sin(\delta\varphi). \tag{3.40}$$

As clearly explained by Orlando (1991), the first Josephson relation still holds in presence of magnetic and electric fields, provided we still work with the gauge-invariant phase.

Assume now that we establish a voltage difference V among the junction's leads. Since we work with metals, the voltage will be approximately uniform along each lead. We then expect a flux difference on both sides of the insulator $\delta\phi(t) = \phi(d/2) - \phi(-d/2) \simeq \int_0^t V(\tau)\mathrm{d}\tau$. Using the connection between flux and phase (3.22), we obtain the second Josephson relation:

$$\delta\varphi = \frac{2\pi}{\Phi_0}\frac{dV}{dt}. \tag{3.41}$$

The Josephson relations combine into an equation connecting flux and current:

$$I = I_c \sin\left(\frac{2\pi}{\Phi_0}\delta\phi\right). \tag{3.42}$$

The dc Josephson effect is a consequence of this equation: a constant voltage bias V induces an oscillating current due to the linearly growing flux:

$$I(t) = I_c \sin\left(\frac{2\pi}{\Phi_0}Vt + \delta\varphi(0)\right). \tag{3.43}$$

However, (3.42) has more general implications, as a constituent equation, allowing us to include Josephson junctions in the general theory of superconducting circuits from Chapter 4. For instance, we can derive the inductive energy stored in the junction as we switch on the voltage – and the flux – to a finite value:

$$E = \int_0^t I(t)V(t)\mathrm{d}t. \tag{3.44}$$

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With the change of variables $V(t)\mathrm{dt} = \mathrm{d}\phi$, we can substitute the expression (3.42) and integrate between the initial and final value of the flux

$$E = \int_0^{\delta\phi} I_c \sin\left(\frac{2\pi}{\Phi_0}\phi\right) \mathrm{d}\phi = -\frac{I_c \Phi_0}{2\pi} \cos\left(\frac{2\pi}{\Phi_0}\delta\phi\right). \tag{3.45}$$

From the point of view of superconducting circuit theory, the junction behaves as a nonlinear inductive element. The inductance may be derived from the current–voltage relation or from an expansion of the inductive energy just derived. In both cases, we obtain a similar expression:

$$L_J = \frac{V}{\frac{\mathrm{d}I}{\mathrm{d}t}} = \frac{\Phi_0}{2\pi} \frac{1}{I_c \cos(2\pi\delta\phi/\Phi_0)}. \tag{3.46}$$

This nonlinear inductance and the constituent equation (3.42) will be repeatedly used in Chapter 4 when studying superconducting qubits, dc-SQUIDs, and rf-SQUIDs.